

Section 1.3. Mathematical Models

Mathematical Models are the use of mathematical terms to describe the behavior of some real-life system or a phenomenon.

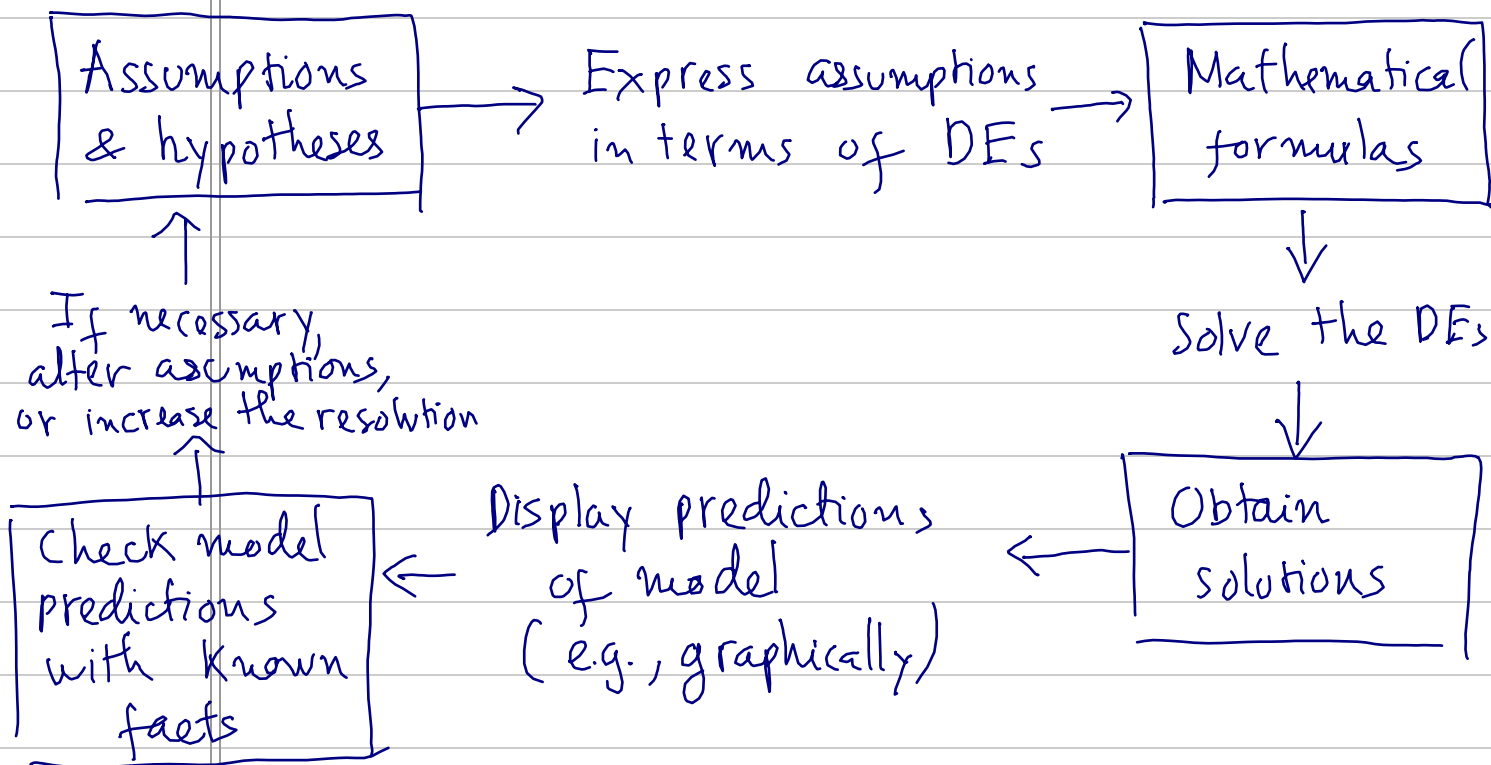
For example, we may wish to understand the mechanism of certain ecosystem by studying the growth of animal populations in that system, or we want to predict which candidate will win in an election, or predict the behavior of stockmarket, or predict the weather etc.

Construction of a mathematical model of a system starts with

- (i) Identification of variables that are responsible for the changing of the system.

We may choose not to incorporate all these variables into the model at first. In this step we are specifying the **level of resolution** of the model.

- (ii) Make a set of reasonable assumptions, or hypotheses, about the system we are trying to describe. The assumptions will also include any empirical laws that may be applicable to the system.



How to make a model with DEs.

POPULATION DYNAMICS.

One of the earliest attempts to model human population growth was by Thomas Malthus (1766-1834) in 1798.

The idea is the assumption that the rate at which the population of a country grows at a certain time is proportional to the total population of the country at that time.

$$\frac{dP}{dt} = KP. \quad (1)$$

It is still used to model growth of small population over short interval of time.

RADIOACTIVE DECAY.

The nucleus of an atom consists of combinations of protons and neutrons. Many of these combinations are unstable - that is, the atom's decay or transmute into atoms of another substance. Such nuclei are said to be radioactive. If we denote $A(t)$ the substance remaining at time t , then the model of radioactive decay is based on the following eq:

$$\frac{dA}{dt} = kA \quad (k < 0).$$

NEWTON'S LAW OF COOLING/WARMING.

$$\frac{dT}{dt} = k(T - T_m), \quad (3)$$

where $T = T(t)$ is the temperature at time t , and T_m is the temperature of surrounding medium. If T_m is a constant, then it stands to reason that $k < 0$.

SPREAD OF DISEASE.

A contagious disease is spread throughout a community by people coming into contact with other people. Denote by $x(t)$ the number of people who have contracted the disease and $y(t)$ the number of people

who have not yet been exposed. It seems reasonable to assume that the rate dx/dt at which the disease spreads is proportional to the encounters between two groups of people:

$$\frac{dx}{dt} = kxy. \quad (4)$$

Suppose a small community has a fixed population of n people. If s infected people are introduced into the community, then $x+y = n+s$, and (4) yields

$$\frac{dx}{dt} = kx(n+s-x) \quad (5)$$

An obvious initial condition accompanying equation (5) is $x(0)=1$.

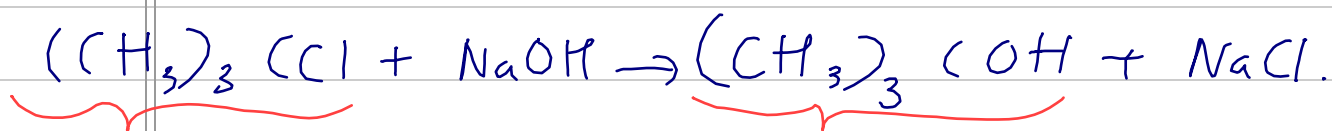
CHEMICAL REACTIONS.

The disintegration of a radioactive substance, governed by the diff. eq. (2), is said to be a **first-order reaction**. In chemistry a few reactions follow in the same law: If the molecules of substance A decompose into smaller molecules, it is natural to assume that the rate at which this decomposition takes place is proportional to the amount of the first substance that has not undergone conversion.

It means that if $x(t)$ is the amount of substance A remaining at any time, then

$$\frac{dx}{dt} = kx \quad (k < 0),$$

For example,



t-butyl chloride

t-butyl alcohol

Only the concentration of the t-butyl chloride controls the rate of reaction here. slower in the reaction



one molecule of sodium hydroxide, NaOH, is consumed for every molecule of methyl chloride, CH_3Cl , thus forming one molecule of methyl alcohol, CH_3OH , and one molecule of sodium chloride, NaCl. In this case the rate of reaction proceeds is proportional to the product of the remaining concentrations of CH_3Cl and NaOH.

To describe the second reaction, let us suppose one molecule of a substance A combines with one molecule of a substance B to form one molecule of a substance C. If x denotes the amount of chemical C formed at time t , and if α and β are the amounts of the two A and B at $t=0$, then the instantaneous amounts of A and B not converted to C are $\alpha-x$ and $\beta-x$, respectively. Hence the rate of formation of C is given by

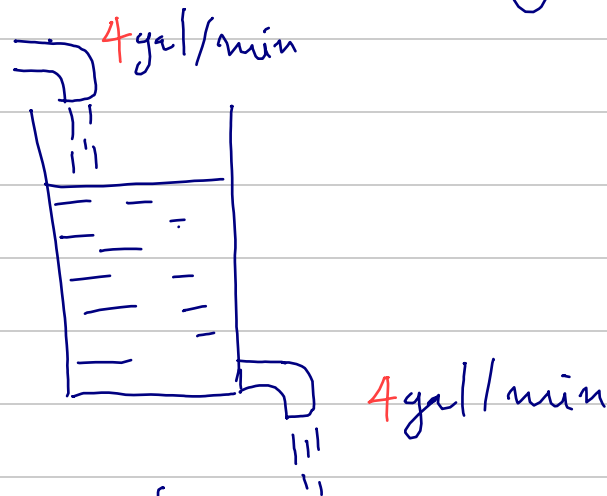
$$\frac{dx}{dt} = k(\alpha-x)(\beta-x) \quad (6)$$

This is called a **second-order reaction**.

MIXTURES.

The mixing of two salt solutions of different concentrations gives rise to a first-order diff. eq. for the amount of salt contained in the mixture.

We have a tank of 200 gallons of brine (i.e. water in which a certain amount of salt has been dissolved). Another brine solution is pumped into the tank at a rate of 4 gallons per minute; the concentration of the salt in this inflow is 3 pounds per gallon. When the solution in the tank is well stirred it is pumped out at the same rate as the entering solution (4 gal/min).



If $A(t)$ denotes the amount of salt in the tank at time t , then

$$\begin{aligned}\frac{dA}{dt} &= \left(\begin{array}{c} \text{input rate} \\ \text{of salt} \end{array} \right) - \left(\begin{array}{c} \text{output rate} \\ \text{of salt} \end{array} \right) \\ &= R_{in} - R_{out} \quad (7)\end{aligned}$$

We have R_{in} is a constant :

$$R_{in} = \underbrace{(3 \text{ lb/gal})}_{\text{concentration}} \underbrace{(4 \text{ gal/min})}_{\text{input rate of brine}} = 12 \text{ lb/min.}$$

While R_{out} is given by

$$R_{out} = \underbrace{\left(\frac{A(t)}{200} \text{ lb/gal}\right)}_{\text{Concentration}} \underbrace{(4 \text{ gal/min})}_{\text{output rate of brine}} = \frac{A(t)}{50} \text{ (lb/min)}$$

Thus (7) becomes

$$\frac{dA}{dt} = 12 - \frac{A}{50} \quad \text{or}$$

$$\frac{dA}{dt} + \frac{A}{50} = 12 \quad (8)$$

DRAINING TANK (reading exercise)

Torricelli's law $v = \sqrt{2gh}$

where the tank is filled to a depth h , g is the acceleration due to gravity.

$$\frac{dV}{dt} = -A_h \sqrt{2gh} \quad (9)$$

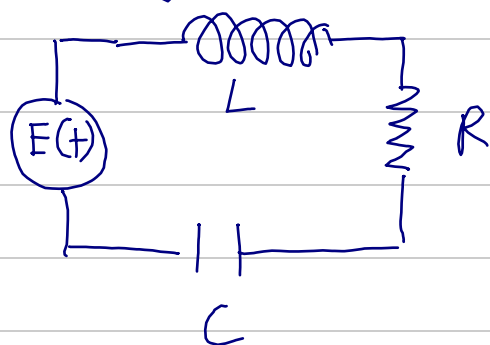
where $V(t)$ denotes volume of water in the tank at time t , A_h is the area of the holes.

We have $V(t) = A_w h$, where A_w is the area of the upper surface. Thus

$$dh/dt = -A_h/A_w \sqrt{2gh} \quad (10).$$

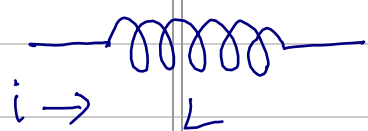
SERIES OF CIRCUITS.

Consider the single-loop LRC-series circuit as in figure

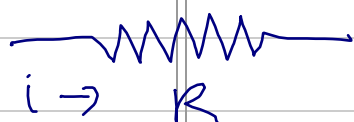


The current in a circuit after a switch is closed is $i(t)$; the charge on a capacitor at time t is denoted by $q(t)$. The letter L, R, C indicate inductance, resistance, and capacitance, respectively, and are constants. By **Kirchhoff's second law**, the impressed voltage $E(t)$ on a closed loop must equal the sum of voltage drops in the loop.

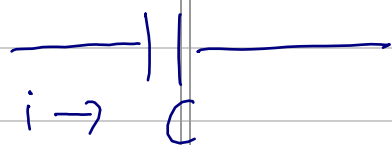
Inductor. inductance L : henries (h/
voltage drop across: $L \frac{di}{dt}$



Resistor. resistance R : ohms (Ω)
voltage drop across: $i R$.



Capacitor. capacitance C : farads (f)
voltage drop across: $\frac{1}{C} q$.



Thus

$$\begin{aligned} E(t) &= L \frac{di}{dt} + i R + \frac{1}{C} q \\ &= L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C} q \end{aligned} \quad \left(i = \frac{dq}{dt} \right) \quad (11)$$

FALLING BODIES.

Newton's first law of motion: "A body either will remain at rest or will continue to move with a constant velocity unless acted on by an external force".

$\Rightarrow$ If the sum of forces $\overset{F}{\downarrow}$ acting on the body is 0, then the acceleration a of the body is 0.

Newton's second law of motion: $F = ma$.

Suppose we toss a coin or a rock from the roof of a tower. Denote by $s(t)$ the position of the coin relative to the ground at time t .

$$\rightarrow F = ma = -mg \quad (\text{gravity force})$$

$$\Rightarrow a = \boxed{\frac{d^2 s}{dt^2} = -g} \quad (12)$$

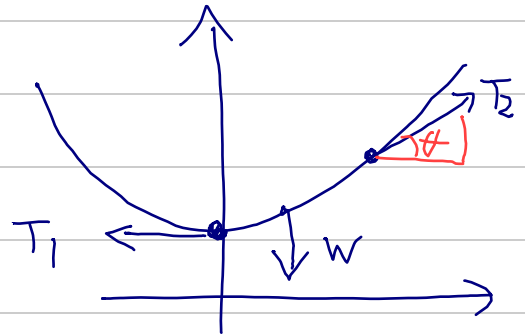
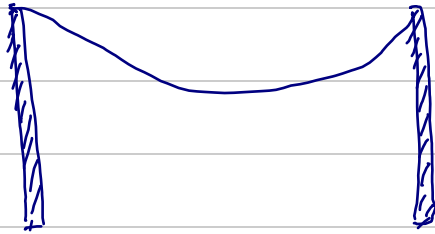
FALLING BODIES & AIR RESISTANCE.

$$F = ma = m \frac{dv}{dt} = mg - Kv \quad (14)$$

$$\Rightarrow \boxed{m \frac{d^2 s}{dt^2} + K \frac{ds}{dt} = mg} \quad (15)$$

SUSPENDED CABLES.

Suppose a flexible cable, wire, or rope is suspended between two vertical supports.



We examine only a portion of the cable between its lowest point P_1 and any point P_2 . There are three forces acting on this cable: the tensions T_1 and T_2 that are tangent to the cable at P_1 and P_2 , and the portion W of the total vertical load between P_1 and P_2 .

We have:

$$T_1 = T_2 \cos \theta \quad \text{and} \quad W = T_2 \sin \theta$$

$\Rightarrow$

$$\tan \theta = \frac{W}{T_1}$$

But $dy/dx = \tan \theta$, we have

$$\frac{dy}{dx} = \frac{W}{T_1} \quad (16)$$